Exact Solutions of the two-dimensional Schrödinger equation with certain central potentials

Shi-Hai Dong*

Physical and Theoretical Chemistry Laboratory, University of Oxford, Oxford OX1 3QZ, UK

and Department of Physics, Cardwell Hall, Kansas State University, Manhattan, Kansas 66506

Abstract

By applying an ansatz to the eigenfunction, an exact closed form solution of the Schrödinger equation in 2D is obtained with the potentials, $V(r) = ar^2 + br^4 + cr^6$, $V(r) = ar + br^2 + cr^{-1}$ and $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$, respectively. The restrictions on the parameters of the given potential and the angular momentum m are obtained.

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*Electronic address: dongsh@phys. ksu. edu

1. Introduction

One of the important tasks of quantum mechanics is to solve the Schrödinger equation with the physical potentials. It is well known that the exact solution of the Schrödinger equation are possible only for the certain potentials such as Coulomb, harmonic oscillator potentials. Some approximation methods are frequently used to obtain the solution. In the past several decades, many efforts have been produced in the literature to study the stationary Schrödinger equation with the central potentials containing negative powers of the radial coordinate [1-31]. Generally, most of authors carried out these problems in the three-dimensional space. Recently, the study of higher order central potentials has been much more desirable to physicists and mathematicians, who want to understand a few newly discovered physical phenomena such as structural phase transitions [1], polaron formation in solids [2] and the concept of false vacuo in filed theory [3]. Besides, the solution of the Schrödinger equation with the sextic potential $V(r) = ar^2 + br^4 + cr^6$ can be applied in the field of fibre optics [4], where one wants to solve a similar problem of an inhomogeneous spherical or circular wave guide with refractive index profile function of the sextic-type potential. Its solution is also applicable to molecular physics [5]. The study for the mixed potential $V(r) = a_1 r + b_1 r^2 + c_1 r^{-1}$ (harmonic+linear+Coulomb) as a phenomenological potential can be used in nuclear physics. However, the study for the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$ has been widely used in the different fields such as atomic physics and optical physics [29-31]. Actually, interest in these anharmonic oscillator-like interactions stems from the fact that the study of the relevant Schrödinger equation, for example, in the atomic and molecular physics as well as nuclear physics , provides us with insight into the physical problem in question.

With the wide interest in the lower-dimensional field theory in the recent literature, however, it is necessary to study the two-dimensional Schrödinger equation with the certain central potentials such as the sextic and mixed potentials as well as the singular even-power potential, an investigation which, to our knowledge, has never been appeared in the literature. Furthermore, two-dimensional models are often applied

to make the more involved higher-dimensional systems tractable. Therefore, it seems reasonable to study the two-dimensional Schrödinger equation with these potentials, which is the purpose of this paper. On the other hand, we have succeeded in studying the two-dimensional Schrödinger equation with some anharmonic potentials [16, 17].

This paper is organized as follows. Section 2 studies the solution of the two-dimensional Schrödinger equation with the sextic potential $V(r) = ar^2 + br^4 + cr^6$ using an ansatz for the eigenfunction. The study for the mixed potential $V(r) = a_1 r + b_1 r^2 + c_1 r^{-1}$ will be presented in section 3. In section 4, we will study the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$. A brief conclusion will be given in the last section 5.

2. The Study for the Sextic Potential

Throughout this paper the natural units $\hbar = 1$ and $\mu = 1/2$ are employed. Consider the two-dimensional Schrödinger equation with a potential $V(r)$ that depends only on the distance r from the origin

$$H\psi = -\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\psi + V(r)\psi = E\psi, \tag{1}$$

where the potential is taken as

$$V(r) = ar^2 + br^4 + cr^6. \tag{2}$$

The choice of r, φ coordinates reflects a model where the full Hilbert space is the tensor product of the space of square-integrable functions on the positive half-line with the space of square-integrable functions on the circle. We therefore write

$$\psi(\mathbf{r}, \varphi) = r^{-1/2} R_m(r) e^{\pm i m \varphi}, \qquad m = 0, 1, 2, \ldots \tag{3}$$

and this factorization leads to a second-order equation for the radial function $R_m(r)$ with vanishing coefficient of the first derivative, i. e.

$$\frac{d^2 R_m(r)}{dr^2} + \left[E - V(r) - \frac{m^2 - 1/4}{r^2}\right] R_m(r) = 0, \tag{4}$$

where m and E denote the angular momentum and energy, respectively. For the solution of Eq. (4), we make an ansatz [6-21] for the radial wave function

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{2n+\delta}, \tag{5}$$

where

$$p_m(r) = \frac{1}{2}\alpha r^2 + \frac{1}{4}\beta r^4. \tag{6}$$

Substituting Eq. (5) into Eq. (4) and equating the coefficient of $r^{2n+\delta+2}$ to zero, one can obtain

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{7}$$

where

$$A_n = \alpha^2 + (3 + 2\delta + 4n)\beta - a \tag{8a}$$

$$B_n = E + (1 + 2\delta + 4n)\alpha \tag{8b}$$

$$C_n = (\delta + 2n)(-1 + \delta + 2n) - (m^2 - 1/4) \tag{8c}$$

and

$$\beta^2 = c \tag{9a}$$

$$2\alpha\beta = b. \tag{9b}$$

It is easy to obtain the values of parameters for $p_m(r)$ from the Eq. (9) written as

$$\beta = \pm\sqrt{c}, \quad \alpha = \frac{b}{2\beta}. \tag{10}$$

If the first non-vanishing coefficient $a_0 \neq 0$ in Eq. (7), and so we can obtain $C_0 = 0$ from Eq. (8c), i. e. $\delta = -m + 1/2$ or $m + 1/2$. In order to retain the well-behaved solution at the origin and at infinity, we choose δ and β as follows:

$$\delta = m + 1/2, \quad \beta = -\sqrt{c}, \tag{11a}$$

from which, one can obtain

$$\alpha = -\frac{b}{2\sqrt{c}}, \tag{11b}$$

On the other hand, if the pth non-vanishing coefficient $a_p \neq 0$, but $a_{p+1} = a_{p+2} = a_{p+3} = \cdots = 0$, it is easy to obtain $A_p = 0$ from Eq. (8a), i. e.

$$a + 2\sqrt{c}(2 + m + 2p) - \frac{b^2}{4c} = 0, \tag{12}$$

which is a restriction on the parameters a, b, c of the potential and angular momentum m and p ($p \leq n$). As we know, A_n, B_n and C_n must satisfy the determinant relation for a nontrivial solution

$$\det \begin{vmatrix} B_0 & C_1 & \cdots & \cdots & \cdots & 0 \\ A_0 & B_1 & C_2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{p-1} & B_p \end{vmatrix} = 0. \tag{13}$$

In order to expound this method, we will give the exact solutions for the different $p = 0, 1$ as follows.

(1): when $p = 0$, it is easy to obtain $B_0 = 0$ from Eq. (13), which, together with Eq. (11), leads to

$$E_0 = \frac{b(1+m)}{\sqrt{c}}. \tag{14}$$

In this case, however, the restriction on the parameters of the potential and the angular momentum m will be obtained as

$$a + 2\sqrt{c}(2 + m) - \frac{b^2}{4c} = 0. \tag{15}$$

The corresponding eigenfunction for $p = 0$ can now be read as

$$R_m^{(0)} = a_0 r^\delta \exp\left[-\frac{b}{4\sqrt{c}} r^2 - \frac{\sqrt{c}}{4} r^4\right], \tag{16}$$

where a_0 is the normalization constant and δ is given by Eq. (11).

(2): when $p = 1$, one can arrive at the following relation from Eq. (13),

$$B_0 B_1 - A_0 C_1 = 0 \tag{17}$$

from which we can obtain

$$E_1 = \frac{b(2+m)}{\sqrt{c}} \pm \frac{\sqrt{b^2(2+m) - 4c(1+m)(2 + 2\sqrt{c}(2+m))}}{\sqrt{c}}. \tag{18}$$

However, the corresponding restriction on the parameters and m can be obtained as

$$a + 2(4+m)\sqrt{c} - \frac{b^2}{4c} = 0. \tag{19}$$

. The corresponding eigenfunction for $p = 1$ can be read as

$$R_m^{(1)} = (a_0 + a_1 r^2) r^\delta \exp\left(-\frac{b}{4\sqrt{c}} r^2 - \frac{\sqrt{c}}{4} r^4\right), \tag{20}$$

where δ has been given by Eq. (11), the coefficients a_0 and a_1 can be determined by the normalization condition completely.

Following this way, we can generate a class of exact solutions through setting $p = 1, 2, \cdots$,etc. For the general case, if the pth non-vanishing coefficient $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, from which we can obtain $A_p = 0$, i. e.

$$\alpha^2 + (3 + 2\delta + 4p) = a. \tag{21}$$

The corresponding eigenfunction can be written as

$$R_m^{(p)} = (a_0 + a_1 r^2 + \cdots + a_p r^{2p}) r^\delta \exp\left[-\frac{b}{4\sqrt{c}} r^2 - \frac{\sqrt{c}}{4} r^4\right], \tag{22}$$

where δ has been given by Eq. (11a), and $a_i (i = 1, 2, \cdots p)$, can be expressed by recurrence relation Eq. (7) and in principle obtained by the normalization condition.

3. The Study for the Mixed Potential

The study for this potential is similar to that for sextic potential except for taking the ansatz as

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{n+\delta}, \tag{23}$$

where p_m is taken as

$$p_m(r) = \alpha r + \frac{1}{2}\beta r^2. \tag{24}$$

We can solve the two-dimensional Schrödinger equation with the this potential

$$V(r) = ar + br^2 + \frac{c}{r}. \tag{25}$$

Similarly, we can obtain the following sets of equations after substituting Eq. (23) into Eq. (4) and equating the coefficients of $r^{\delta+n}$ to zero,

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0 \tag{26}$$

where

$$A_n = E + \beta(1 + 2n + 2\delta) \tag{27a}$$

$$B_n = -c + \alpha(2n + 2\delta) \tag{27b}$$

$$C_n = (n + \delta)(-1 + n + \delta) - (m^2 - 1/4) \tag{27c}$$

and

$$\beta^2 = b, \qquad 2\alpha\beta = a. \tag{27d}$$

Similar to the above choices, we can choose β and δ as $-\sqrt{b}$ and $m + 1/2$, respectively. According to these choices, the parameter α can be obtained as

$$\alpha = -\frac{a}{2\sqrt{b}}. \tag{28}$$

Now, let us consider the case $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, then we can get $A_p = 0$. In this case, the energy eigenvalue can be written as

$$E_p = 2\sqrt{b}(1 + m + p). \tag{29}$$

Likewise, the nontrivial solution of recursion relation Eq. (26) can be obtained by Eq. (13). The exact solutions for $p = 0$ and $p = 1$ can be discussed below.

(1): when $p = 0$, we can arrive at

$$E_0 = 2\sqrt{b}(1 + m) \tag{30}$$

.

and $B_0 = 0$, i. e.

$$2c\sqrt{b} = a(1 + 2m), \tag{31}$$

which is a restriction on the corresponding parameters of the potential and the angular momentum m. The eigenfunction, however, can be read as

$$R_m^{(0)} = a_0 r^\delta \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right], \tag{32}$$

where δ is taken as $m + 1/2$, the coefficient a_0 can be evaluated by the normalization condition.

(2): when $p = 1$, the energy eigenvalue can be written as

$$E_1 = 2\sqrt{b}(2 + m). \tag{33}$$

Moreover, we can obtain the restriction on the parameters of the potential and the angular momentum m from the determinant relation Eq. (13) as $B_0 B_1 = A_0 C_1$, i. e.

$$\left\{c + \frac{(1+2m)a}{2\sqrt{b}}\right\}\left\{c + \frac{(3+2m)a}{2\sqrt{b}}\right\} = 2\sqrt{b}(1 + 2m). \tag{34}$$

In this case, the corresponding eigenfunction can be written as

$$R_m^{(1)} = (a_0 + a_1 r) r^{\delta} \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right], \tag{35}$$

where a_0 and a_1 can be obtained by the recursion relation Eq. (26) and the normalization relation.

Similarly, if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, we can get $A_p = 0$. In this case, the eigenfunction can be written as

$$R_m^{(p)} = (a_0 + a_1 r + \cdots + a_p r^p) r^{\delta} \exp\left[-\frac{ar + br^2}{2\sqrt{b}}\right], \tag{36}$$

where δ is taken as $m + 1/2$, and the coefficients $a_i (i = 1, 2, \cdots, p)$ can be calculated by Eq. (26) and the normalization condition.

4. The Study for the Singular Even-Power Potential

Similar to above discussion, the study for the central singular even-power potential can be taken the following ansatz

$$R_m(r) = \exp[p_m(r)] \sum_{n=0} a_n r^{2n+\delta}, \tag{37}$$

where p_m is taken as

$$p_m(r) = \frac{1}{2}\alpha r^2 + \frac{1}{2}\beta r^{-2}. \tag{38}$$

We can solve the two-dimensional Schrödinger equation with the potential

$$V(r) = ar^2 + \frac{b}{r^2} + \frac{c}{r^4} + \frac{d}{r^6}. \tag{39}$$

Likewise, one can get the following sets of equations after substituting the ansatz Eq. (37) into Eq. (4) and equating the coefficients of $r^{\delta+n}$ to zero,

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2} a_{n+2} = 0, \tag{40}$$

where

$$A_n = E + \alpha(1 + 2\delta + 4n) \tag{41a}$$

$$B_n = -b - 2\alpha\beta - (m^2 - 1/4) + (\delta + 2n)(-1 + \delta + 2n) \tag{41b}$$

$$C_n = (3 - 2\delta - 4n) - c \tag{41c}$$

and

$$\alpha^2 = a, \qquad \beta^2 = d. \tag{42}$$

Similar to the above choices, we can choose α and β as $-\sqrt{a}$ and $-\sqrt{d}$, respectively. Moreover, if $a_0 \neq 0$, then one can obtain $C_0 = 0$, i. e.

$$\delta = (3/2 + \mu), \tag{43}$$

where $\mu \equiv \frac{c}{2\sqrt{d}}$. However, if $a_p \neq 0$, but $a_{p+1} = a_{p+2} = \cdots = 0$, then it leads to $A_p = 0$, from which one can obtain the energy eigenvalue as

$$E_p = \sqrt{a}(4 + 4p + 2\mu). \tag{44}$$

In the following section, let us discuss the corresponding exact solutions for $p = 0$ and $p = 1$.

(1): when $p = 0$, we can arrive at

$$E_0 = \sqrt{a}(4 + 2\mu). \tag{45}$$

In this case, it means that $B_0 = 0$ from the determinant relation Eq. (13), which will lead to the constraint condition between the parameters of the potential and the angular momentum quantum number m,

$$(1 + \mu)^2 - b - 2\sqrt{ad} - m^2 = 0. \tag{46}$$

The corresponding eigenfunction, however, can be written as

$$R_m^{(0)} = a_0 r^{\delta} \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right], \tag{47}$$

where and hereafter δ is given by Eq. (43) and the coefficient a_0 can be obtained by the normalization condition.

(2): $p = 1$, the energy eigenvalue can be obtained from Eq. (44) as follows

$$E_1 = \sqrt{a}(8 + 2\mu). \tag{48}$$

In this case, the determinant relation Eq. (13) means that $B_0 B_1 = A_0 C_1$, which will result in the following restriction on the parameters and angular momentum quantum m,

$$\left[-b - 2\sqrt{ad} + (1+\mu)^2 - m^2\right]\left[-b - 2\sqrt{ad} + (3+\mu)^2 - m^2\right] - 16\sqrt{ad} = 0. \tag{49}$$

The eigenfunction for $p = 1$ can be read as

$$R_m^{(1)} = (a_0 + a_1 r^2) r^{\delta} \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right], \tag{50}$$

where and hereafter δ is given by Eq. (43), and $a_i (i = 0, 1)$ can be calculated from Eq. (40) and the normalization relation. Following this method, we can obtain a class of exact solutions through setting the different p. Generally, the corresponding eigenfunction for p can be written as

$$R_m^{(p)} = (a_0 + a_1 r + \cdots + a_p r^{2p}) r^{\delta} \exp\left[-\frac{\sqrt{a}r^2 + \sqrt{d}r^{-2}}{2}\right], \tag{51}$$

where $a_i (i = 0, 1, \cdots, p)$ can be evaluated from recursion relation Eq. (40) and the normalization condition.

5. Concluding Remarks

In this paper, applying an ansatz to the eigenfunction, we have obtained the exact solutions of the two-dimensional Schrödinger equation with the certain potentials such as the sextic potential $V(r) = ar^2 + br^4 + cr^6$, the mixed potential $V(r) = ar + br^2 + cr^{-1}$

as well as the singular even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$, respectively. The corresponding restrictions on the parameters of the potential and the angular momentum m have been obtained for the different potentials. The study for other classes of certain central potentials by this method is in progress.

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